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Convergence to translating solutions for a class of quasilinear parabolic boundary problems

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1.

Huisken [Hu] proved that graphs in dimensions $n \geqq 2$ evolving by the non-parametric mean curvature flow with vertical contact angle converge to constant functions. In this short note, we show that solutions to a class of quasilinear equations in one dimension with fixed angle of contact to the boundary converge to translating solitons (see illustration).

Let $\Omega_d \subset R^2$ with $d \in R^+$ be the rectangular region

$$\Omega_d = \{x \geqq 0\} \cap \{x \leqq d\} \quad \text{with} \quad \partial\Omega = \partial\Omega_0 \cup \partial\Omega_d = \{\{x = 0\} \cup \{x = d\}\}.$$

We then consider the quasilinear parabolic equation on Ω_d given by $u_t = (v(u_x))_x$ where $v \in C^\infty(R^1)$ and $v' > 0$. The angles $\alpha_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $(i = 0, d)$ will be measured counter-clockwise from the x-axis. Solutions may be kept in a bounded region by rescaling with translation constant

$$A(\alpha_0, \alpha_d, d) = \frac{v(\tan\alpha_d) - v(\tan\alpha_0)}{d}. \tag{1.1}$$

1.1. Theorem. *The boundary value problem*

$$\begin{cases} u_t - (v(u_x))_x = 0 & \text{in } \Omega_d \times [0, \infty) \\ u_x(i, t) = \tan\alpha_i & \text{on } \partial\Omega_i \times [0, \infty), i = 0, d \\ u(\cdot, 0) = u_0 & u_0 \in C^\infty(\bar{\Omega}_d) \end{cases} \tag{1.2}$$

converges as $t \to \infty$ to a solution moving by translation with speed $A(\alpha_0, \alpha_d, d)$. In particular, if $\alpha_d = \alpha_0$, the graph determined by $u(x, t)$ converges to a straight line.

1.2. Corollary (standard heat flow). *For $v(u_x) = u_x$, the graph of $u(x, t)$ converges as $t \to \infty$ to a uniquely determined portion of a parabola or a straight line.*

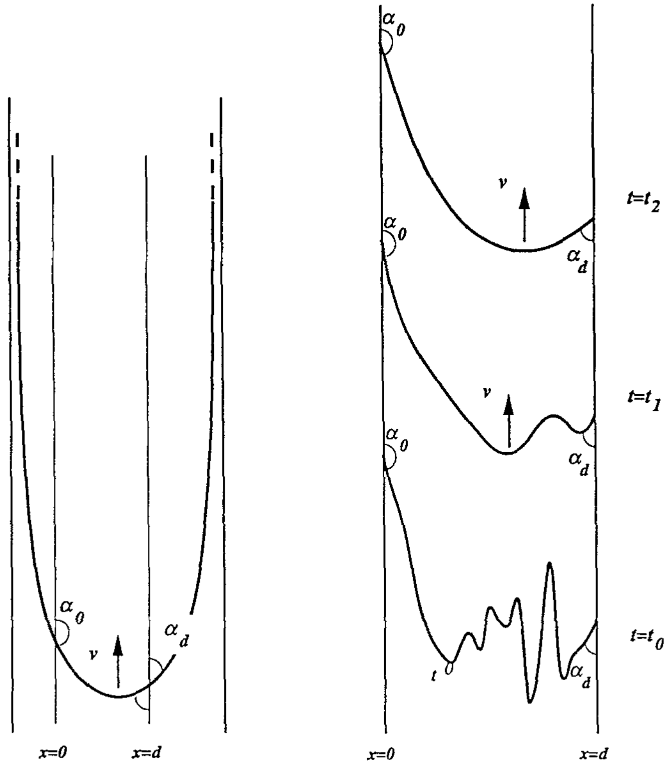

Fig. 1 *Portion of a Translating Solution* *Convergence*

1.3. Corollary (nonparametric mean curvature flow). *For* $v(u_x) = \arctan(u_x)$, *the graph of* $u(x,t)$ *converges as* $t \to \infty$ *to a uniquely determined portion of the Grim Reaper or a straight line.*

2.

In this section we will denote $v = v(u_x)$ and $v' = v'(u_x) = \dfrac{d}{du_x} v(u_x)$. From $u_t = (v)_x = v' u_{xx}$, we may compute:

$$\frac{\partial}{\partial t} u_x = v'(u_x)_{xx} + v'' u_{xx}^2 ; \tag{2.1}$$

$$\frac{\partial}{\partial t} v = v' v_{xx} . \tag{2.2}$$

The angle conditions which were imposed on u_x and time derivatives of these conditions yield:

$$\begin{cases} v(u_x(i,t)) = v(\tan \alpha_i) & \text{for } i = 0, d \\ v_t(u_x(i,t)) = v_{tt}(u_x(i,t)) = v_{ttt}(u_x(i,t)) = \ldots = 0 & \text{for } i = 0, d \end{cases} \tag{2.3}$$

We will now establish the following results (see, for example, [LSU, GT] as references).

2.1. Lemma. *For $t \in [0, \infty]$, then we have the a priori estimates:*

(a) $|u_x|(x,t) \leqq \sup |u_x|(x,0)$;

(b) $c_1 \geqq v'(t) \geqq c_2 > 0$ *with some positive* c_1, c_2 *only depending on* $u_x(x,0)$;

(c) *moreover,* v, v', v'', v''' *and all higher derivatives have uniform bounds depending only on* $u_x(x,0)$.

Proof. Standard arguments [LSU] can be applied directly to (1.1). In fact, long time estimates may also be established by using the integral estimates which appear later in this paper. Q.E.D.

The problem now is to show convergence. For this, we will use integral estimates (see also [GH] for similar arguments). For notational simplicity, we will always use the constant c to denote constants depending only on $u_x(x,0)$.

2.2. Lemma. *For all* $\varepsilon \in R^+$, *there exists a time* T *such that for* $t \geqq T$ *the integral* $\int v_t^2 dx(t) \leqq \varepsilon$.

Proof. Note that Lemma 2.1(b) implies that a bound on $\int v_t^2 dx$ is equivalent to a bound on $\int v_{xx}^2 dx$. We now wish to show that $\int v_{xx}^2 dx$ is frequently small:

$$\frac{d}{dt}\int v_x^2 dx = 2\int v_x v_{tx} dx = -2\int v_{xx} v_t dx$$

$$= -2\int v' v_{xx}^2 dx < 0\,. \tag{2.4}$$

That is, $\exists c$ such that $\int\limits_0^\infty \int v_{xx}^2 dx dt \leqq c < \infty$. Now we show that the integral cannot have arbitrarily small spikes in time. In fact, it has at most exponential growth:

$$\frac{d}{dt}\int v_{xx}^2 dx = 2\int v_{xx} v_{txx} dx = -2\int v_{xxx} v_{tx} dx = -2\int v_{xxx}(v' v_{xx})_x\, dx$$

$$= -2\int \left(v_{xxx} v_{xxx} v' + \frac{v''}{v'}\, v_{xxx} v_{xx} v_x \right) dx$$

$$\leqq -2\int v' \left(v_{xxx} + \frac{v''}{2v'^2}\, v_{xx} v_x \right)^2 - \frac{v''^2}{4v'^3}\, v_{xx}^2 v_x^2 dx$$

$$\leqq 2\int \frac{v''^2}{4v'^3}\, v_{xx}^2 v_x^2 dx$$

$$\leqq c\int v_{xx}^2 dx\,. \tag{2.5}$$

So, $\int v_{xx}^2 dx \to 0$ as $t \to \infty$. Hence $\int v_t^2 dx \to 0$ as $t \to \infty$. Q.E.D.

We will make use of the following "Poincaré"-type estimates.

2.3. Lemma (basic integral estimates).

$$-\int |v_{xx}| dx + A(\alpha_0, \alpha_d, d) \leqq v_x \leqq \int |v_{xx}| dx + A(\alpha_0, \alpha_d, d) \tag{2.6}$$

$$\sup \left(\frac{\partial^k v}{\partial t^k} \right)^2 \leqq d \int \left| \frac{\partial^k v_x}{\partial t^k} \right|^2 dx \quad \textit{for } k \geqq 1 \tag{2.7}$$

$$\sup \left(\frac{\partial^k v_x}{\partial t^k} \right)^2 \leqq d \int \left| \frac{\partial^k v_{xx}}{\partial t^k} \right|^2 dx \quad \textit{for } k \geqq 1 \tag{2.8}$$

Proof. The first inequality (2.6) follows from the estimate

$$\inf g - \sup g + \int g dx \Big/ \int dx \leqq g \leqq \sup g - \inf g + \int g dx \Big/ \int dx . \tag{2.9}$$

Now, (2.7) follows from the fact that $\frac{\partial^k}{\partial t^k} v(i,t) = 0$ $(i = 0, d)$ and (2.8) follows from (2.9) by integrating by parts. That is, $\int \frac{\partial^k}{\partial t^k} v_x dx = 0$. Q.E.D.

2.4. Lemma. *There exist constants* $c_1, c_2 \in R^+$ *such that*

$$\int v_{tx}^2 dx(t) \leqq c_1 e^{-c_2 t} \int v_{tx}^2 dx(0) .$$

Proof. We first compute the evolution equation:

$$\begin{aligned}
\frac{d}{dt} \int v_{tx}^2 dx &= 2 \int v_{tx} v_{ttx} dx = -2 \int v_{txx} v_{tt} dx = -2 \int v_{xxt} v_{tt} dx \\
&= -2 \int \left(\frac{v_t}{v'}\right)_t v_{tt} dx = -2 \int \frac{v_{tt}^2}{v'} - \frac{v'' v_{tt} v_t^2}{v'^3} dx \\
&\leqq - \int \frac{v_{tt}^2}{v'} dx + c \int v_t^4 dx
\end{aligned}$$

where $c \in R^+$ is some constant. Estimate (2.8) with $k = 1$ may be modified by the comparison

$$v_{txx}^2 \leqq 2\left(\frac{v_{tt}}{v'}\right)^2 + 2\left(\frac{v'' v_t^2}{v'^3}\right)^2$$

to give

$$\int v_{tx}^2 dx \leqq 2d^2 \int \left(\frac{v_{tt}}{v'}\right)^2 dx + 2d^2 \int \left(\frac{v'' v_t^2}{v'^3}\right)^2 dx .$$

Hence, there exist constants $c_1, c_2 \in R$, such that:

$$\frac{d}{dt} \int v_{tx}^2 dx \leqq -c_1 d^{-2} \int v_{tx}^2 dx + c_2 \int v_t^4 dx .$$

Lemma 2.2, tells us that $\int v_t^2 dx$ may be made as small as we want after a sufficiently long time. Therefore, using the inequality $\sup(v_t)^2 \leqq d \int v_{tx}^2 dx$ and choosing ε sufficiently small, we obtain

$$\begin{aligned}
\frac{d}{dt} \int v_{tx}^2 dx &\leqq -c_1 d^{-2} \int v_{tx}^2 dx + c_2 \sup(v_t)^2 \int v_t^2 dx \\
&\leqq -c_1 d^{-2} \int v_{tx}^2 dx + \varepsilon c_2 \sup(v_t)^2 \\
&\leqq -c_3 d^{-2} \int v_{tx}^2 dx .
\end{aligned}$$

Thus, after a waiting period determined by the previous lemma, the integral $\int v_{tx}^2 dx$ decays exponentially to zero. Q.E.D.

Main theorem. *Now, formula* (2.7) *with* $k = 1$ *and Lemma 2.4 imply that* $v_t \to 0$ *exponentially as* $t \to \infty$. *Therefore, by formula* (2.6), $v_x \to A(\alpha_0, \alpha_d, d)$ *and* $u_t = v_x \to 0$ *exponentially as* $t \to \infty$. *The fact that higher derivatives converge exponentially to zero is obtained by a standard induction argument.* Q.E.D.

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Note added in proof. We have recently generalized the results here and in [Hu] to the case of surfaces. [Altschuler, S., Wu., L.-F.: Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle. Preprint (1992)]