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Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle

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Abstract. In this work, we study surfaces over convex regions in $\mathbb{R}^2$ which are evolving by the mean curvature flow. Here, we specify the angle of contact of the surface to the boundary cylinder. We prove that solutions converge to ones moving only by translation.

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1. Introduction

In this paper, we evolve graphs defined over compact, convex domains $\Omega \subset \mathrm{R}^2$ by the nonparametric mean curvature flow with prescribed contact angle to $\partial\Omega$. After obtaining a uniform gradient estimate, we show that the solutions converge to solutions which move at a constant speed of translation. Since the angle of contact and curvature of the boundary are allowed to be non-constant, one may construct translating solutions which are neither convex nor rotationally symmetric.

This work complements both the work of Huisken [H] where the problem in dimensions $n \geq 2$ with vertical contact angle was solved and work by the authors [AW] where the problem for one-dimensional graphs evolving by a general class of quasilinear equations with prescribed contact angle was solved. In contrast to Huisken's integral methods, our gradient bound is derived from pointwise methods and the maximum principle.

Let $\Omega \subset \mathrm{R}^2$ be a compact domain with smooth boundary $\partial\Omega$. Let $k > 0$ be the curvature of $\partial\Omega$. The inward pointing normal and counterclockwise tangent vector to $\partial\Omega$ will be denoted by N and T. The upward normal for a graph $u: \Omega \to \mathrm{R}^1$ is $\gamma = (-Du, 1)/\sqrt{1+|Du|^2}$. The angle of contact between the graph and the boundary, $\alpha: \partial\Omega \to (0, \pi)$, is given by $\langle \gamma, N \rangle = \cos\alpha$ or $D_{\mathrm{N}} u = -\cos\alpha\sqrt{1+|Du|^2}$.

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1.1. IVP. *The nonparametric mean curvature flow with specified contact angle is*

$$\begin{cases} u_t = \left(\delta_{ij} - \dfrac{D_i u D_j u}{1+|Du|^2}\right) D_i D_j u & \text{on } \mathrm{Q_T} \\ D_{\mathrm{N}} u = -\cos\alpha \sqrt{1+|Du|^2} & \text{on } \Gamma_{\mathrm{T}} \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \Omega_0 \end{cases} \tag{1.1}$$

where $u_0 \in C^\infty(\bar{\Omega})$, $D_{\mathrm{N}} u_0 = -\cos\alpha\sqrt{1+|Du_0|^2}$ *on* $\partial\Omega_0$, *and*

$$\mathrm{Q_T} = \Omega \times [0, \mathrm{T}]; \quad \Gamma_{\mathrm{T}} = \partial\Omega \times [0, \mathrm{T}]; \quad \Omega_t = \Omega \times \{t\}.$$

We then prove the following results. Note that $\sin\alpha > 0$ since $\alpha \in (0, \pi)$.

1.2. Theorem. *If* $\exists k_0, \alpha_0, \delta_0 \in \mathrm{R}^+$ *such that*

$$k - |D_{\mathrm{T}}\alpha| \geq \delta_0 > 0; \quad k_0 \geq k > 0; \quad |\alpha| \leq \alpha_0 < \pi, \tag{1.2}$$

then, for solutions to IVP 1.1.:
1. $\exists c_1 = c_1(\alpha_0, \delta_0, k_0, u_0) > 0$ *so that* $|Du|^2 \leq c_1$ *on* Q_∞, *thus* $u(x,t) \in C^\infty(\mathrm{Q}_\infty)$*;*
2. $u(x,t)$ *converges as* $t \to \infty$ *to a surface* u_∞ *(unique up to translation) which moves at a constant speed* C *(uniquely determined by the boundary data);*
3. *if* $\int_{\partial\Omega} \cos\alpha = 0$ *then* $C = 0$, *hence* u_∞ *is a minimal surface.*

There is much interest in studying solutions of the mean curvature flow which move by self-similarity. Such solutions often model the asymptotic behaviour of developing singularities [Alt; An; Hu2]. For curves, so-called "Grim Reaper" graph given by $y = \log\sec x$ moves by translation under the flow. In dimension 2, numerical evidence has been given for the existence of complete surfaces embedded in R^3 which move by translation (see [S]). An easy consequence of Theorem 1.2 is the following.

1.3. Corollary. *There exists a complete, noncompact, rotationally symmetric solution of the mean curvature flow which moves only by translation.*

We remark that Hamilton [Ha] has recently shown that any strictly convex solution of the Mean Curvature flow existing for all time and having an interior maximum of the curvature must be a translating soliton.

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2. Estimates

Notation. It is often convenient to use the following notation.

$$\begin{aligned} v &= \sqrt{1+|Du|^2}, \\ a^{ij} &= \delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}, \\ \Delta_a &= a^{ij} D_i D_j, \\ u_t &= \frac{\partial u}{\partial t}. \end{aligned}$$

Also, for vectors, V, W or matrices A, B we will use the shorthand

$$\langle V, W\rangle_a = a^{ij}V_iW_j\,; \quad \langle V, W\rangle_\delta = \delta^{ij}V_iW_j\,; \quad \langle A, B\rangle_{a,\delta} = a^{ij}\delta^{kl}A_{ik}B_{jl}\,.$$

On $\partial\Omega$, we define $a^{TN} = a^{ij}T_iN_j$, $a^{TT} = a^{ij}T_iT_j$, and $a^{NN} = a^{ij}N_iN_j$. Note, we will make use of the formula $D_VD_Wu = V^iW^jD^2_{ij}u + \langle D_VW, Du\rangle$.

The boundary

We first give a smooth extension of N and T (defined in Sect. 1) to a thin neighborhood of $\partial\Omega$. Consider the set of coordinate vectors $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right\}$ where $\frac{\partial}{\partial r}$ is orthogonal to the level sets of the distance function to $\partial\Omega$, $\left\langle\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right\rangle = 0$, $\left|\frac{\partial}{\partial r}\right|^2 = 1$, and $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right\}_{\partial\Omega} = \{\mathrm{N}, \mathrm{T}\}$. Define the function ϕ such that $\left|\phi^{-1}\frac{\partial}{\partial\theta}\right|^2 = 1$. We now define the extended normal and tangent vectors to be the orthonormal frame $\{\mathrm{N}, \mathrm{T}\} = \left\{\frac{\partial}{\partial r}, \phi^{-1}\frac{\partial}{\partial\theta}\right\}$.

The following lemma will be useful.

2.1. Lemma. *On $\partial\Omega$, we have the following identities:*
1. *for $f \in C^\infty(\bar\Omega)$, the interchange of derivatives is given by $D_\mathrm{N}D_\mathrm{T}f = D_\mathrm{T}D_\mathrm{N}f + kD_\mathrm{T}f$;*
2. *$D_\mathrm{T}T = kN$; $D_\mathrm{T}N = -kT$; $D_\mathrm{N}T = D_\mathrm{N}N = 0$.*

Proof. At $\partial\Omega$, the Frenet formula gives $D_\mathrm{T}T = kN$ and $D_\mathrm{T}N = -kT$. By differentiating $\phi^2 = \left\langle\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right\rangle$ in the direction of the inward normal, an easy computation gives

$$\phi\frac{\partial\phi}{\partial r} = -\phi^2 k\,.$$

Hence,

$$\begin{aligned} D_\mathrm{N}D_\mathrm{T}f &= \frac{\partial}{\partial r}\left(\phi^{-1}\frac{\partial f}{\partial\theta}\right) = -\phi^2\frac{\partial\phi}{\partial r}\frac{\partial f}{\partial\theta} + \phi^{-1}\frac{\partial}{\partial\theta}\frac{\partial f}{\partial r} \\ &= k\phi^{-1}\frac{\partial f}{\partial\theta} + \phi^{-1}\frac{\partial}{\partial\theta}\frac{\partial f}{\partial r} = kD_\mathrm{T}f + D_\mathrm{T}D_\mathrm{N}f\,. \end{aligned}$$

It only remains to prove the assertion $D_\mathrm{N}T = D_\mathrm{N}N = 0$. In coordinates, the above commutator formula implies that, for any f,

$$N^iT^jD^2_{ij}f + \langle D_\mathrm{N}T, Df\rangle = +T^iN^jD^2_{ij}f + \langle D_\mathrm{T}N, Df\rangle + kD_\mathrm{T}f$$

and, switching the order of differentiation,

$$\langle D_\mathrm{N}T, Df\rangle = \langle D_\mathrm{T}N, Df\rangle + kD_\mathrm{T}f\,.$$

Now $D_\mathrm{N}T = 0$ since $D_\mathrm{T}N = -kT$. Also, $D_\mathrm{N}N = 0$ follows from $\langle D_\mathrm{N}N, N\rangle = 0$ and from $\langle D_\mathrm{N}T, N\rangle + \langle T, D_\mathrm{N}N\rangle = 0$. Q.E.D.

Now, the reader may verify that (2.1) implies (2.2) and (2.3)

$$D_{\mathrm{N}}u = -\cos\alpha\sqrt{1+|Du|^2}\,, \tag{2.1}$$

$$|D_{\mathrm{N}}u|^2 = \cot^2\alpha(1+|D_{\mathrm{T}}u|^2)\,, \tag{2.2}$$

$$|D_{\mathrm{T}}u|^2 \sin^2\alpha(1+|Du|^2) - 1\,. \tag{2.3}$$

By differentiating conditions (2.1)–(2.3) in the time and tangential directions, all the derivatives of u on $\partial\Omega$, except $D_{\mathrm{N}}D_{\mathrm{N}}u$, can be expressed in terms of the first derivatives of u. The useful computations are

$$D_{\mathrm{N}}u_t = -\cos\alpha\frac{Du\cdot Du_t}{\sqrt{1+|Du|^2}}\,, \tag{2.4}$$

$$D_{\mathrm{T}}D_{\mathrm{N}}u = \sin\alpha(D_{\mathrm{T}}\alpha)v - \cos\alpha(D_{\mathrm{T}}v)\,, \tag{2.5}$$

$$D_{\mathrm{N}}D_{\mathrm{T}}u = \sin\alpha(D_{\mathrm{T}}\alpha)v - \cos\alpha(D_{\mathrm{T}}v) + kD_{\mathrm{T}}u\,, \tag{2.6}$$

$$D_{\mathrm{T}}D_{\mathrm{T}}u = \frac{\sin\alpha\cos\alpha(D_{\mathrm{T}}\alpha)v^2 + \sin^2\alpha v D_{\mathrm{T}}v}{D_{\mathrm{T}}u}\,. \tag{2.7}$$

Existence of solutions

The existence for small time and uniqueness of solutions to IVP 1.1 follow from the linear theory [LDU] and the inverse function theorem (see [S]). We assume that smooth solutions exist on the time interval $[0,\mathrm{T}]$. That is, on the time interval $[0,\mathrm{T}]$, all derivatives of u have (possibly large) bounds. Below, we will establish a time independent a priori bound on the gradient of the solution. This will turn the quasilinear evolution equation into a uniformly parabolic equation. The higher order regularity follows from standard theory and the infinite time existence of smooth solutions follows.

Controlling the time derivative

Using the maximum principle, we establish an a priori bound on $|u_t|^2$.

2.2. Lemma. $\sup_{\mathrm{Q_T}}|u_t|^2 = \sup_{\Omega_0}|u_t|^2$. *That is,* $\exists c_0 = c_0(u_0)\in\mathrm{R}^+$ *such that* $\forall(x,t)\in\mathrm{Q_T}$

$$|u_t|^2(x,t)\le c_0\,.$$

Proof. We first show that the maximum must occur on $\Gamma_{\mathrm{T}}\cup\Omega_0$. Let $(a^{ij})'$ be the differential of $a^{ij} = a^{ij}(x,u,Du) = a^{ij}(x,z,p)$ with respect to p. It is then a simple computation to show

$$\frac{\partial}{\partial t}|u_t|^2 = \Delta_a|u_t|^2 - 2\langle Du_t, Du_t\rangle_a + D_iD_ju\langle(a^{ij})', \nabla|u_t|^2\rangle_\delta$$

where $D_iD_ju\langle(a^{ij})'\cdot\nabla|u_t|^2\rangle = \sum_{k,i,j=1}^{n} D_iD_ju\left(\frac{\partial a^{ij}}{\partial p^k}\nabla_k|u_t|^2\right)$. Since all of the coefficients are bounded in $\mathrm{Q_T}$, one may apply the weak maximum principle to obtain $\sup_{\mathrm{Q_T}}|u_t|^2 = \sup_{\Gamma_{\mathrm{T}}\cup\Omega_0}|u_t|^2$.

Next the possibility that the maximum occurs at $(\xi, \tau) \in \Gamma_\mathrm{T}$ is explored. Now, if $\max_{\Omega_\tau} |u_t|^2 = |u_t|^2(\xi, \tau) > 0$ then $(D_\mathrm{T} u_t)(\xi, \tau) = 0$. From (2.4)

$$\begin{aligned}(D_\mathrm{N} u_t)(\xi, \tau) &= -\cos\alpha \frac{D_\mathrm{N} u D_\mathrm{N} u_t + D_\mathrm{T} u D_\mathrm{T} u_t}{\sqrt{1+|Du|^2}}(\xi, \tau) \\ &= -\cos\alpha \frac{D_\mathrm{N} u D_\mathrm{N} u_t}{\sqrt{1+|Du|^2}}(\xi, \tau) \\ &= \cos^2\alpha (D_\mathrm{N} u_t)(\xi, \tau).\end{aligned}$$

That is, $\sin^2\alpha(D_\mathrm{N} u_t)(\xi, \tau) = 0$. Since $\alpha \in (0, \pi)$, we may conclude $(D_\mathrm{N} u_t)(\xi, \tau) = 0$. Therefore, by the Hopf Lemma, $\frac{\partial}{\partial t}|u_t|^2(\xi, \tau) \le 0$ and the lemma is proven. Q.E.D.

We note that the above result actually holds true for nonconvex domains $\Omega \subset \mathrm{R}^n$.

Controlling the gradient

This is the crucial step in establishing the infinite time existence of solutions to IVP 1.1. As in the previous estimate, we will make strong use of the boundary algebra.

2.3. Theorem. *Under the assumptions* (1.2), *then* $\exists c_1 = c_1(\alpha_0, \delta_0, k_0, u_0)$ *such that*

$$\sup_{\mathrm{Q_T}} |Du|^2 \le c_1 .$$

Proof. Again, we first show that the maximum must occur on $\Gamma_\mathrm{T} \cup \Omega_0$. The evolution equation for the full gradient is

$$\frac{\partial}{\partial t}|Du|^2 = \Delta_a |Du|^2 - 2|DDu|^2_{a,\delta} - \left\langle \frac{D|Du|^2}{1+|Du|^2}, D|Du|^2 \right\rangle_a$$

and, since all of the coefficients are bounded in $\mathrm{Q_T}$, the weak maximum principle implies that $\sup_{\mathrm{Q_T}} |Du|^2 = \sup_{\Gamma_\mathrm{T} \cup \Omega_0} |Du|^2$.

Next we assume that the maximum of $|Du|^2$ occurs at $(\xi, \tau) \in \Gamma_\mathrm{T}$. There are two cases. First, if $|D_\mathrm{T} u|^2(\xi, \tau) < 1$, then from $\sin^2\alpha(1+|Du|^2) = 1 + |D_\mathrm{T} u|^2$ we see that

$$|Du|^2(\xi, \tau) < \frac{2}{\sin^2\alpha_0} - 1 \tag{2.8}$$

and the bound is established.

Otherwise, we may assume that $|D_\mathrm{T} u|^2(\xi, \tau) \ge 1$. At (ξ, τ),

$$\begin{aligned}D_\mathrm{N}|Du|^2(\xi, \tau) &\le 0, \\ D_\mathrm{T}|Du|^2(\xi, \tau) &= 0 = D_\mathrm{T} v(\xi, \tau).\end{aligned}$$

Hence (2.5)–(2.7) simplify and become at (ξ, τ):

$$D_\mathrm{T} D_\mathrm{N} u = \sin\alpha (D_\mathrm{T}\alpha) v, \tag{2.9}$$

$$D_\mathrm{N} D_\mathrm{T} u = \sin\alpha (D_\mathrm{T}\alpha) v + k D_\mathrm{T} u, \tag{2.10}$$

$$D_\mathrm{T} D_\mathrm{T} u = \frac{\sin\alpha\cos\alpha (D_\mathrm{T}\alpha) v^2}{D_\mathrm{T} u}. \tag{2.11}$$

Now our trick is to turn the following expression into one which contains only first derivatives:

$$(D_N u)(D_N D_N u)(\xi,\tau) + (D_T u)(D_N D_T u)(\xi,\tau) \le 0 . \tag{2.12}$$

We accomplish this by the expedient of rewriting u_t. Using the fact that (ξ,τ) is a maximum, (2.1)–(2.3) and (2.9)–(2.11) yield at (ξ,τ):

$$\begin{aligned}
u_t &= a^{TT} D_T D_T u +^{TN} D_T D_N u + a^{NT} D_N D_T u + a^{NN} D_N D_N u \\
&\quad - a^{TT}\langle D_T T, Du\rangle - a^{TN}\langle D_T N, Du\rangle - a^{NT}\langle D_N T, Du\rangle - a^{NN}\langle D_N N, Du\rangle \\
&= \frac{1+|D_N u|^2}{1+|Du|^2}\,\frac{\sin\alpha\cos\alpha(D_T\alpha)v^2}{D_T u} - \frac{D_T u D_N u}{1+|Du|^2}\sin\alpha(D_T\alpha)v \\
&\quad - \frac{D_N u D_T u}{1+|Du|^2}(\sin\alpha(D_T\alpha)v + kD_T u) + \frac{1+|D_T u|^2}{1+|Du|^2} D_N D_N u \\
&\quad - k\,\frac{1+|D_N u|^2}{1+|Du|^2} D_N u - k\,\frac{D_T u D_N u}{1+|Du|^2} D_T u \\
&= \left(\frac{\cos\alpha\sin\alpha(D_T\alpha)}{D_T u}\right)(v^2 + |D_T u|^2) + \frac{2k\cos\alpha|D_T u|^2}{v} \\
&\quad + \sin^2\alpha D_N D_N u + k\cos\alpha\,\frac{1+|D_N u|^2}{v} .
\end{aligned}$$

That is,

$$\begin{aligned}
\sin^2\alpha D_N D_N u &= u_t - \left(\frac{\cos\alpha\sin\alpha(D_T\alpha)}{D_T u}\right)(v^2+|D_T u|^2) \\
&\quad - \frac{2k\cos\alpha|D_T u|^2}{v} - \frac{k\cos\alpha(1+|D_N u|^2)}{v} .
\end{aligned}$$

Now, using (2.10) and substituting the expression above for the double normal derivative into (2.12), we obtain

$$\begin{aligned}
&-\cos\alpha v u_t + \cos^2\alpha\sin\alpha(D_T\alpha)v\left(\frac{v^2+|D_T u|^2}{D_T u}\right) \\
&\quad + 2k|D_T u|^2\cos^2\alpha + k\cos^2\alpha(1+|D_N u|^2) \\
&\quad + \sin^2\alpha(\sin\alpha(D_T\alpha)v + kD_T u)D_T u \le 0 .
\end{aligned} \tag{2.13}$$

We leave it to the reader to verify the following algebraic identities:

$$2k\cos^2\alpha|D_T u|^2 + k\cos^2\alpha(1+|D_N u|^2) + k\sin^2\alpha|D_T u|^2 = k(v^2-1) ,$$

$$\frac{\cos^2\alpha\sin(D_T\alpha)}{D_T u}(v^2+|D_T u|^2)v + \sin^3\alpha(D_T\alpha)(D_T u)v = \frac{\sin\alpha(D_T\alpha)}{D_T u}v(v^2-1) ,$$

$$v^2 - 1 = \frac{|D_T u|^2}{\sin^2\alpha} + \frac{\cos^2\alpha}{\sin^2\alpha} .$$

Now, we may simplify (2.13) to give

$$k(v^2-1) + \frac{D_T u}{\sin\alpha}(D_T\alpha)v \le (\cos\alpha)v u_t - \frac{\cos^2\alpha}{\sin\alpha D_T u}(D_T\alpha)v . \tag{2.14}$$

From (2.3) we see that

$$\left|\frac{D_T u}{\sin\alpha}\right| \le v .$$

Thus, using $|D_T u| \ge 1$ and $v \ge 1$, we may estimate (2.14) as

$$(k - |D_T\alpha|)v \le |\cos\alpha|\,|u_t| + \frac{|D_T\alpha|}{\sin\alpha} + k . \tag{2.15}$$

By assumption (1.2), $k - |D_T\alpha| \ge \delta_0 > 0$. By Lemma 2.2 $|u_t|^2 \le c_0$. Therefore

$$v \le \delta_0^{-1}\left(c_0^{1/2} + k + \frac{k}{\sin\alpha}\right) . \quad \text{Q.E.D.}$$

Elliptic interlude

We wish to point out now that the above gradient estimate also may be used to solve the elliptic version of the problem.

2.4. BVP. *The elliptic version of IVP 1.1 is*

$$\begin{cases} a^{ij}(Dw)D_iD_jw = C & \text{on } \Omega \\ D_N w = -\cos\alpha\sqrt{1+|Dw|^2} & \text{on } \partial\Omega , \end{cases} \tag{2.16}$$

where C is a uniquely determined constant.

Since the operator in (2.16) may be written as $\sqrt{1+|Dw|^2}D_i(D_iw/\sqrt{1+|Dw|^2})$, one may integrate by parts to see

$$C = \frac{\int_{\partial\Omega} \cos\alpha\, ds}{\int_{\Omega} (1+|Dw|^2)^{-1/2}\, dx} . \tag{2.17}$$

One method for solving this boundary value problem is to solve the following problem

2.5. BVP.

$$\begin{cases} a^{ij}(Dw_\varepsilon)D_iD_jw_\varepsilon = \varepsilon w_\varepsilon & \text{on } \Omega \\ D_N w_\varepsilon = -\cos\alpha\sqrt{1+|Dw\varepsilon|^2} & \text{on } \partial\Omega . \end{cases} \tag{2.18}$$

2.6. Theorem. *Under assumptions* (1.2) *a unique, smooth solution to BVP 2.4 exists.*

Proof. It is well known that solutions to this problem exist for $\varepsilon > 0$. If we can show that $\exists c_0$, independent of ε, such that $|\varepsilon w_\varepsilon|^2 \le c_0$, then we may replace u_t with $\varepsilon w_\varepsilon$ in the gradient estimate of Theorem 2.3 and conclude that a limit solution exists for $\varepsilon \to 0$.

The following argument was shown to us by Urbas [U]. Let ψ be a smooth function on Ω satisfying $D_N\psi < -\cos\alpha\sqrt{1+|D\psi|^2}$ on $\partial\Omega$. For example, let d be the distance function to $\partial\Omega$ and let A be a constant such that $A < -\cos\alpha(\cdot)\sqrt{1+A^2}$ on $\partial\Omega$. Then a function ψ defined to be $\psi = Ad$ near the boundary and extended to be a smooth function on all of Ω would satisfy our requirements. In any case, for some such choice of ψ, let $\xi \in \Omega$ be a point where $\psi - w_\varepsilon$ has its minimum.

If $\xi \in \partial\Omega$ then $D_T\psi(\xi) = D_T w_\varepsilon(\xi)$ and $D_N\psi(\xi) \geq D_N w_\varepsilon(\xi)$. Now, for a fixed constant a, the monotonicity in q of the function $\frac{q}{\sqrt{1+a^2+q^2}}$ gives

$$-\cos\alpha(\xi) > \frac{D_N\psi}{\sqrt{1+|D\psi|^2}}(\xi) \geq \frac{D_N w_\varepsilon}{\sqrt{1+|Dw_\varepsilon|^2}}(\xi) = -\cos\alpha(\xi)$$

which is a contradiction.

Thus $\xi \in \Omega$ and $D\psi(\xi) = Dw_\varepsilon(\xi)$ and $D^2\psi(\xi) \geq D^2 w_\varepsilon(\xi)$. Therefore, there exists a constant $c = c(\psi)$ such that

$$c \geq a^{ij}(D\psi)D_iD_j\psi(\xi) \geq a^{ij}(Dw_\varepsilon)D_iD_jw_\varepsilon(\xi) = \varepsilon w_\varepsilon(\xi).$$

Hence $\varepsilon\psi(\cdot) - \varepsilon w_\varepsilon(\cdot) \geq \varepsilon\psi(\xi) - \varepsilon w_\varepsilon(\xi)$ implies

$$\varepsilon w_\varepsilon(\cdot) \leq \varepsilon\psi(\cdot) - \varepsilon\psi(\xi) + c.$$

A similar barrier argument bounds $\varepsilon w_\varepsilon$ from below. Then, as in [LTU], $|Dw_\varepsilon|^2 \leq c_1$ implies $|D(\varepsilon w_\varepsilon)|^2 \to 0$ as $\varepsilon \to 0$ and we may conclude that $\varepsilon w_\varepsilon \to C$.

Now, assume that there exist two solutions w_1 & w_2 solving (2.16) with C_1 & C_2 on the right hand side and $C_1 < C_2$. Without loss of generality, we may assume that $w_1 \geq w_2$. Then $w = w_1 - w_2$ satisfies a linear elliptic differential inequality $L(w) < 0$ and it follows from the maximum principle that the minimum of w must occur at $\xi \in \partial\Omega$. Then $|D_T w_1|^2(\xi) = |D_T w_2|^2(\xi) = a^2$ for some $a \in \mathrm{R}^+$. Since both solutions satisfy the same boundary conditions,

$$\frac{D_N w_1}{\sqrt{1+a^2+|D_N w_1|^2}}(\xi) = \frac{D_N w_2}{\sqrt{1+a^2+|D_N w_2|^2}}(\xi).$$

Again, we may use the strict monotonicity in q of the function $\frac{q}{\sqrt{1+a^2+q^2}}$ to conclude $D_N w_1 = D_N w_2$. But $D_N w = 0$ yields a contradiction to the Hopf boundary point lemma. Thus, $C_1 \geq C_2$.

By reversing the roles of w_1 and w_2 we may obtain the opposite inequality. Thus, $C_1 = C_2$.

The proof that $w_1 = w_2$ follows along similar lines. Q.E.D.

For a solution $w = w(x)$ of BVP 2.4, it is obvious that $\tilde{w}(x,t) = w(x) - Ct$ solves the parabolic problem. That is, $\tilde{w}$ is a solution that just translates upwards with speed C. This allows us to obtain an oscillation bound on solutions to the parabolic problem.

2.7. Corollary. *For a solution $u = u(x,t)$ of IVP 1.1, $\exists c_2 \in \mathrm{R}^+$ such that*

$$|u(x,t) - Ct| \leq c_2.$$

Proof. This result follows by sandwiching the parabolic solution between two translating elliptic solutions and applying a maximum principle similar to the one used in Theorem 2.6. Q.E.D.

3. Consequences

We now show the uniqueness of limit solutions to the parabolic IVP. We wish to thank Gerhand Huisken for showing us the following application of the strong maximum principle.

3.1. Theorem. *Let u_1 and u_2 be any two solutions of IVP 1.1 and let $u = u_1 - u_2$. Then u becomes a constant function as $t \to \infty$. In particular, since $\tilde{w} = Ct + w$ solves IVP 1.1 when w solves BVP 2.4, all limit solutions of IVP 1.1 are $\tilde{w}$ up to translation.*

Proof. As in the proof of Theorem 2.6, we now see that u satisfies a linear parabolic equation

$$\begin{cases} \frac{\partial}{\partial t} u = \tilde{a}^{ij} D_{ij} u + \tilde{b}^i D_i u & \text{on } Q_T \\ 0 = \tilde{c}^{ij} D_i u N_j & \text{on } \Gamma_T \end{cases}, \tag{3.1}$$

where

$$\tilde{a}^{ij} = \int_0^1 a^{ij}(\theta D u_1 + (1-\theta) D u_2) d\theta$$

and $\tilde{b}^i, \tilde{c}^{ij}$ are determined similarly (see [GT]). Note that $\tilde{c}^{ij}$ is a positive definite matrix. The strong maximum principle implies that $osc(t) = \max u(\cdot, t) - \min u(\cdot, t) \geq 0$ is a strictly decreasing function in time unless u is constant. Now, if $\lim_{t\to\infty} u(\cdot, t)$ were not a constant function, then a limit of $u_n(\cdot, t) = u(\cdot, t - t_n)$ as $t_n \to \infty$ would yield a solution on $\Omega \times (-\infty, +\infty)$ which would not be constant but on which $osc(t)$ would be constant. This, however, would contradict the strong maximum principle. Q.E.D.

It follows that, in the case where the average cosine of the contact angle is zero, solutions converge to minimal surfaces. We wish to highlight this fact with an explicit computation.

3.2. Remark. If $C = 0$ for the elliptic barrier, $\lim_{t\to\infty}(u_t) \to 0$. That is, the solutions converge to the corresponding minimal surfaces.

Proof. First we establish the fact that u_t is frequently small.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega v \mathrm{d}x = \int_\Omega \frac{D_i u_t D_i u}{v} \mathrm{d}x = -\int_\Omega \frac{u_t^2}{v} \mathrm{d}x + \int_{\partial\Omega} u_t \cos\alpha \mathrm{d}s$$

implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega v \mathrm{d}x - \int_{\partial\Omega} u \cos\alpha \mathrm{d}s \right) = -\int_\Omega \frac{u_t^2}{v} \mathrm{d}x \,.$$

Since $\int_{\partial\Omega} \cos\alpha ds = 0$ implies $C = 0$, Theorem 2.3 and Corollary 2.7 imply $\exists c_3 \in \mathrm{R}^+$ such that

$$\int_0^\infty \int_\Omega \frac{u_t^2}{v} \mathrm{d}x \, \mathrm{d}t \leq c_3 \,.$$

One may then apply standard estimates to conclude that $\lim_{t\to\infty} u_t^2 \to 0$. Q.E.D.

Finally, to illustrate an application of our work, we will construct a non-compact translating solution. This solution has been observed numerically by Angenent, Grayson, Huisken, Stone and others by studying a singular ODE.

3.3. Corollary. *There exists a convex, rotationally symmetric, translating solution defined over the plane with any prescribed constant speed C. This solution is "asymptotic" to a paraboloid.*

Prof. We will first work with the stationary problem: consider solving $Hv = C > 0$ with constant boundary angle on the disk D_{r_0} of radius r_0. Note, H denotes the mean curvature of the surface and Hv is another way of expressing the left hand side of (2.16). In polar coordinates (r, θ), the equation $Hv = C$ simplifies to

$$Hv = \frac{u_{rr}}{v^2} + \frac{u_r}{r} = C\,.$$

A consequence of the maximum principle is that there are no interior maxima for v and hence $v^2 = 1 + u_r^2$ grows monotonely from $v(0) = 1$ at the origin to its maximum value at the boundary. Thus, since $Hv = C$, the mean curvature has its maximum value at the origin, $H(0) = C$, and decrease monotonely to the boundary. Hence $H_r \leq 0$.

We now wish to claim on D_{r_0} that for each $C \geq 0$ one may find an α so that BVP 2.4 is solved with exactly the constant C on the right hand side. Note that C depends continuously on α. One may easily see that $\alpha = \pi/2$ gives a solution with $C = 0$. Let us assume that $\exists C_0 > 0$, an upper bound, such that there does not exist an α realizing C_0.

Now, we differentiate $Hv = C$ to otain $H_r v + H v_r = 0$. For all $r \leq r_0$, we have

$$v_r = \frac{u_r u_{rr}}{v} =_r \left(Hv^2 - \frac{u_r}{r}\, v\right)$$

and so

$$-H_r v = H u_r \left(Hv^2 - \frac{u_r}{r}\, v\right) = C u_r \left(C - \frac{u_r}{r}\right). \tag{3.2}$$

At the boundary, $u_r^2 = \cot^2(\alpha)$. By letting $\alpha \to 0$, we can make $u_r \to +\infty$ at $\partial\Omega$. However, it an upper bound C_0 existed, from (3.1) it may be seen that we can choose u_r so large at the boundary that $H_r(r_0) > 0$ which is a contradiction.

Let us fix some $C > 0$. We may then find a sequence of solutions $u_{(n,C)}$ on the disks D_n or radius n which translate at a fixed speed C. We will normalize the family so that $u_{(n,C)}(0,0) = 0$. Now, two radially symmetric solutions, traveling at the same speed, differ only by a translation. (Proof: assume that w_1, w_2 are two solutions with $w = w_1 - w_2$. Since w satisfies a linear elliptic equation $L(w) = 0$, the maximum and minimum values of w occur on $\partial\Omega$. But, rotational symmetry forces the maximum and minimum to be equal, hence w is a constant.)

Hence, on D_{n-1}, the solutions $u_{(n,C)} = u_{(n-1,C)}$. In this way, by letting $n \to \infty$, one may obtain a limit solution $u_{(\infty,C)}$. Clearly, $Ct + u_{(\infty,C)}$ is a solution over R^2 to the parabolic problem.

Note the fact that $H > 0$ is monotone decreasing in r together with higher derivative bounds give $\lim_{r\to\infty} H_r = 0$. Since, for $r > 0$ we have $0 < u_r/v \leq 1$, one may see that (3.1) shows $C = \lim_{r \to \infty} \frac{u_r}{r}$. More precisely $\forall \varepsilon, \exists R$ so that $\forall r \geq R$

$$\frac{(C-\varepsilon)}{2} r^2 \leq u(r) \leq \frac{(C-\varepsilon)}{2} r^2 . \quad \text{Q.E.D.}$$

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